

On the Calabi–Yau Phase of $(0, 2)$ Models

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Abstract

We study the Calabi-Yau phase of a certain class of $(0, 2)$ models. These are conjectured to be equivalent to exact $(0, 2)$ superconformal field theories which have been constructed recently. Using the methods of toric geometry we discuss in a few examples the problem of resolving the singularities of such models and calculate the Euler characteristic of the corresponding gauge bundles.

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1 Introduction

The classical solutions to the perturbative string theory with unbroken $N = 1$ spacetime supersymmetry provide us with the only known string vacua in four dimensions. As is well-known, the $(0, 2)$ superconformal invariance on the string worldsheet together with an integrality condition on the charges of $U(1)$ current in the superconformal algebra are equivalent to $N = 1$ spacetime supersymmetry [1, 2]. Therefore, the $(0, 2)$ superconformal field theories (particularly the $(0, 2)$ Calabi-Yau σ models) seem to be the natural context for the (geometric) string compactification. Yet another source of interest in $(0, 2)$ models is due to the fact that the phenomenological prospects of such models are much more promising than those of, e.g. $(2, 2)$ theories, because they lead to the more realistic gauge groups [3, 4] like $SO(10)$ and $SU(5)$.

Nevertheless, these models have received less attention in the early days of string theory. This was in part because of the assertion made in [5] that the generic $(0, 2)$ Calabi-Yau σ models suffer from destabilization by the worldsheet instantons. (However, the recent work of [6] shows that these models are not destabilized by such nonperturbative σ model effects.) In spite of the early work of [7, 8, 9] the technical difficulty in constructing $(0, 2)$ models remained the other main obstacle in the study of these models.

The Witten's gauged linear sigma model approach [10] has dramatically changed the state of affairs. It provided a powerful tool in constructing $(2, 2)$ and $(0, 2)$ models and in analyzing their 'phase structures'. Using this framework the authors of [11] have constructed and analyzed plenty of $(0, 2)$ models in their 'Landau-Ginzburg phase'. The subsequent works of [13, 14] proposed an identification between an exact $(0, 2)$ SCFT and a certain model of [11]. This exact $(0, 2)$ SCFT was a Gepner type model which was constructed using the simple current methods. Inspired by this proposal the authors of [15, 16] have tried to extend this identification to a larger class of $(0, 2)$ models. The starting point was a general solution of anomaly cancellation condition yielding a large set of consistent σ model data. In these works the attention has been paid primarily to the connection of the exact SCFTs and the Landau-Ginzburg models.

The above mentioned geometric data defining a $(0, 2)$ supersymmetric $U(1)$ gauge theory in general result in a singular Calabi-Yau variety and some stable vector bundle (\simeq locally free sheaf), the so-called gauge bundle (or 'gauge sheaf'), on it. The 'Calabi-Yau phases' correspond to the possible crepant desingularizations of this singular model. As discussed in [18], the $(0, 2)$ singularities have two different origins. One set of them comes from the singularities of the base Calabi-Yau variety and the other one is associated to the singularities of the gauge sheaf. (These are points of the base varieties, where the gauge sheaf fails to be locally free.)

In this paper we are going to study the Calabi-Yau phase of some examples from the

class of models constructed in [15, 17]. We take the same attitude towards this problem as the authors of [18]. After resolving the singularities of the base variety we calculate the Euler characteristic of the ‘pulled back gauge bundles’. The methods which are used here are those of toric geometry, specially the intersection theory and the Riemann-Roch theorem for coherent sheaves on the toric varieties.

In section 2 we briefly review some basic aspects of the gauged linear sigma model approach and give the solution of the anomaly cancellation condition found in [15, 16]. As mentioned above, this general solution provides a large set of consistent data for $(0, 2)$ Calabi-Yau σ models. Using these data we construct the models of interest to us. The next section will be of a technical nature. Here we discuss the relevant mathematical tools from toric geometry. In section 4 we discuss our examples. We conclude with some comments about open problems and directions for future work.

2 The gauged linear sigma models

In this section we explain the basic ideas of the gauged linear sigma model approach without going into details. In doing so we pinpoint those aspects which are crucial for our considerations in the next sections. For more details we refer to [10, 11, 12].

The starting point is a $(0, 2)$ supersymmetric $U(1)$ gauge theory that represents a nonconformal member of the universality class of a $(0, 2)$ superconformal field theory. The action S which describes this model is

$$S = S_g + S_k + S_W + S_{F.I.} \quad , \quad (1)$$

where S_g, S_k are the kinetic parts of the gauge and matter sectors, S_W is the superpotential and $S_{F.I.}$ is the $U(1)$ Fayet-Iliopoulos D-term.

Let $\Phi_i (i = 1, \dots, n+1)$, P be chiral scalar superfields with the $U(1)$ charges $w_i, -m$ and $\Lambda^a (a = 1, \dots, \ell+1)$, Γ be chiral Fermi superfields with the $U(1)$ charges $q_a, -d$. The simplest $(0, 2)$ superpotential that one can write down has the following form

$$S_W = \int d^2z d\theta \quad (\Gamma W(\Phi_i) + P \Lambda^a F_a(\Phi_i)) \quad , \quad (2)$$

where W and F_a are homogeneous polynomials in Φ_i of degree d and $m - q_a$, respectively. Integrating out the D auxiliary field in the gauge multiplet and the auxiliary fields in the chiral Fermi superfields, we get the bosonic potential

$$U = |W(\phi_i)|^2 + |p|^2 \sum_a |F_a(\phi_i)|^2 + \frac{e^2}{2} \left(\sum_i w_i |\phi_i|^2 - m |p|^2 - r \right)^2 \quad (3)$$

where the parameter r is the coefficient in the Fayet-Iliopoulos D-term and ϕ_i, p denote the lowest terms of the superfields Φ_i and P .

Now varying the parameter r this model exhibits different ‘phases’. By minimizing the classical bosonic potential U for large positive r we obtain

$$\sum_i w_i |\phi_i|^2 = r, \quad W(\phi_i) = 0, \quad p = 0. \quad (4)$$

Taking the quotient by the action of the $U(1)$ gauge group these equations describe a Calabi-Yau variety X as the zero locus of the homogeneous polynomial $W(\phi_i)$ in the weighted projective space $\mathbb{P}(w_1, \dots, w_{n+1})$ with the Kähler class proportional to r . The right-moving fermions ψ_i (the superpartners of ϕ_i) couple to the tangent bundle of X which is given by the cohomology of the monad

$$0 \rightarrow \mathcal{O} \rightarrow \bigoplus_{i=1}^{n+1} \mathcal{O}(w_i) \rightarrow \mathcal{O}(d) \rightarrow 0, \quad (5)$$

and in the same way the left-moving fermions λ_a (the lowest components of the superfields Λ_a) couple to the vector bundle V defined by the cohomology of the following monad

$$0 \rightarrow \mathcal{O} \rightarrow \bigoplus_{a=1}^{\ell+1} \mathcal{O}(q_a) \rightarrow \mathcal{O}(m) \rightarrow 0. \quad (6)$$

So we find that our gauged linear sigma model for positive r reduces in the infrared limit to a $(0, 2)$ Calabi-Yau σ model with the target space X , a hypersurface in the weighted projective space $\mathbb{P}(w_1, \dots, w_{n+1})$, and a rank ℓ gauge bundle V on it which is defined by the exact sequence

$$0 \rightarrow V \rightarrow \bigoplus_{a=1}^{\ell+1} \mathcal{O}(q_a) \xrightarrow{F_a} \mathcal{O}(m) \rightarrow 0. \quad (7)$$

But this is not the whole story! Apart from some ‘regularity’ conditions on F_a and q_a [11], these geometric data still have to satisfy an important condition that comes from the cancellation of the $U(1)$ gauge anomaly. Imposing the condition $c_2(V) = c_2(X)$ guarantees this cancellation. This leads, in turn, to the following quadratic Diophantine equation:

$$m^2 - \sum_{a=1}^{\ell+1} q_a^2 = d^2 - \sum_{i=1}^{n+1} w_i^2. \quad (8)$$

Note also that, with the above choice of $U(1)$ charges, the first Chern class of V vanishes

$$\sum_{a=1}^{\ell+1} q_a - m = 0. \quad (9)$$

which guarantees the existence of spinors on it. For large negative values of r the vanishing of the bosonic potential yields

$$\phi_i = 0 \quad (\text{for all } i), \quad |p|^2 = \frac{-r}{m}. \quad (10)$$

In this case p and its superpartner become massive and drop out of the low energy theory. The gauge group $U(1)$ breaks down to the subgroup \mathbb{Z}_m because the charge of p is m . We are therefore left with a $(0, 2)$ Landau-Ginzburg orbifold in the infrared limit. Absorbing the vacuum expectation value of p by a trivial rescaling of the fields we get the superpotential

$$S_w = \int d^2z d\theta \quad (\Gamma W(\Phi_i) + \Lambda^a F_a(\Phi_i)). \quad (11)$$

Summarizing the above discussion we have found that the Calabi-Yau σ models and the Landau-Ginzburg models can be interpreted as two different ‘phases’ of the same underlying theory.

It should be noted that the Calabi-Yau varieties of interest, which have been defined as hypersurfaces in weighted projective spaces are in general singular, whereas their corresponding physical theories are well-behaved. This is an indication of the fact that the strings probe the smooth geometry of the target Calabi-Yau spaces. Therefore, our phase picture of the moduli space of the theory is not complete. To remedy this we have to desingularize our model and consider the moduli space of this new model within which the moduli space of the original model will be embedded.

In the framework of gauged linear sigma models the process of desingularization amounts above all to embedding the original theory in a new one with gauge group $U(1) \times \dots \times U(1)$ (N copies) and $N - 1$ new chiral scalar superfields $\Upsilon_1, \dots, \Upsilon_{N-1}$ and then determine the charges of the fields with respect to the full gauge group. This new model then has N Kähler moduli parameters r_1, \dots, r_N , one for each Fayet-Iliopoulos D-term of the $U(1)$ factors of the new gauge group. Now by varying these parameters and finding the minima of the bosonic potential one can recover as before the phase structure of the moduli space.

As is well-known [29, 31], there is an equivalent formulation of the whole story in terms of toric geometry. It provides us with some efficient computational tools for analyzing the phase structure of the moduli space. The basic idea here is that the relevant information describing a theory is encoded in the combinatorial data of some reflexive polytope Δ in a lattice \mathbf{N} and the phase structure of the theory is then determined by the possible triangulations of this polytope. By the Calabi-Yau phase we now mean a phase which corresponds to a maximal triangulation of Δ . We still have to deal with the other set of data in a $(0, 2)$ model, namely the gauge bundle data. We postpone this issue and the technicalities of resolving the base variety with the methods of toric geometry to the next

section.

Now we give a general solution of anomaly cancellation condition found in [15, 16]. The starting point is the following geometric data that defines a rank 4 stable vector bundle V on a Calabi-Yau hypersurface X in $\mathbb{P}(w_1, \dots, w_5)$:

$$0 \rightarrow V \rightarrow \bigoplus_{a=1}^5 \mathcal{O}(q_a) \xrightarrow{F_a} \mathcal{O}(m) \rightarrow 0 \quad (12)$$

By setting $m = d$ and $\{q_1, \dots, q_5\} = \{w_1, \dots, w_5\}$ the equation (8) is trivially satisfied. Assume that for one of the weights, say w_5 , we have $d/w_5 \in 2\mathbb{Z} + 1$. Replace w_5 by $2w_5$ and define $w_6 := (m - w_5)/2$. Furthermore, take instead of m the new integers $m_1 := m - w_5$ and $m_2 := (m + 3w_5)/2$ into account. One can easily check that $\{w_1, w_2, w_3, w_4, w_5; m\}$ and $\{w_1, w_2, w_3, w_4, 2w_5, w_6; m_1, m_2\}$ satisfy

$$m^2 - \sum_{i=1}^5 w_i^2 = m_1^2 + m_2^2 - \sum_{i=1}^4 w_i^2 - (2w_5)^2 - w_6^2. \quad (13)$$

In [16, 17] this equation has been interpreted as the anomaly cancellation condition for the defining data of a $(0, 2)$ Calabi-Yau σ model with the same gauge bundle as in (12), defined now on a Calabi-Yau complete intersection in $\mathbb{P}(w_1, w_2, w_3, w_4, 2w_5, w_6)$. Contrary to this interpretation we assume that these data describe a $(0, 2)$ supersymmetric $U(1)$ gauge theory whose ‘Calabi-Yau phase’ is determined by the following exact sequence

$$0 \rightarrow V \rightarrow \bigoplus_{i=1}^4 \mathcal{O}(w_i) \oplus \mathcal{O}(2w_5) \oplus \mathcal{O}(w_6) \xrightarrow{F} \mathcal{O}(m_1) \oplus \mathcal{O}(m_2) \rightarrow 0 \quad (14)$$

on the same Calabi-Yau variety X in $\mathbb{P}(w_1, \dots, w_5)$ as before. The choice of (14) is technically more appropriate because we have a good control on the reflexive polytopes in four dimensions. However, it would be interesting to study the case of Calabi-Yau complete intersection and compare the results with those of (14).

3 Intersection ring and Riemann-Roch theorem

In this section we discuss some concepts of toric geometry which provide us with the main tools for the calculations of the next section. For the details of the definitions and constructions used here we refer to the standard works [20, 22, 21]. In the first part of this section we use the homogeneous coordinate ring approach which is more appropriate for our field theoretical considerations. The original motivation for its development was, however, the desire to have a construction of toric variety and related objects similar to those of \mathbb{P}^n in the classical algebraic geometry [23].

To begin with we first introduce some notation. Let \mathbf{N} and $\mathbf{M} = \text{Hom}(\mathbf{N}, \mathbb{Z})$ denote a dual pair of lattices of rank d and $\langle \cdot, \cdot \rangle$ be the canonical pairing on $\mathbf{M} \times \mathbf{N}$. Further, let

$\mathbf{N}_{\mathbb{R}} = \mathbf{N} \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathbf{M}_{\mathbb{R}} = \mathbf{M} \otimes_{\mathbb{Z}} \mathbb{R}$ be the \mathbb{R} -scalar extensions of \mathbf{N} and \mathbf{M} , respectively. $T = \mathbf{N} \otimes_{\mathbb{Z}} \mathbb{C}^* = \text{Hom}_{\mathbb{Z}}(\mathbf{M}, \mathbb{C}^*)$ is the d dimensional algebraic torus which acts on the toric variety \mathbb{P}_{Σ} defined by the (complete simplicial) fan Σ in $\mathbf{N}_{\mathbb{R}}$. For a cone $\sigma \in \Sigma$ the dual cone, σ^{\vee} , is defined as usual by $\sigma^{\vee} = \{m \in \mathbf{M}_{\mathbb{R}} \mid \langle m, n \rangle \geq 0 \text{ for all } n \in \sigma\}$ and $\text{cosp}\sigma^{\vee}$ is the greatest subspace of $\mathbf{M}_{\mathbb{R}}$ contained in σ^{\vee} . The open affine variety in \mathbb{P}_{Σ} associated to σ is denoted by $X_{\sigma^{\vee}}$. Let $\Sigma^{(k)}$ be the set of k dimensional cones in Σ . By e_i we denote the primitive lattice vectors on the one dimensional cones in $\Sigma^{(1)} = \{\rho_1, \dots, \rho_n\}$. This set will play an important role in what follows.

Each one dimensional cone ρ_i defines a T -invariant Weil divisor, denoted by D_i , which is the closed subvariety $X_{\text{cosp}\rho_i^{\vee}}$ in $X_{\rho_i^{\vee}}$. This is indeed the closed T -orbit associated to ρ_i [20]. The finitely generated free abelian group $\bigoplus_{i=1}^n \mathbb{Z} \cdot D_i$ is the group of T -invariant Weil divisors in \mathbb{P}_{Σ} . Each $m \in \mathbf{M}$ gives a character $\chi^m : T \rightarrow \mathbb{C}^*$, and hence χ^m is a rational function on \mathbb{P}_{Σ} . It defines the Cartier divisor $\text{div}(\chi^m) = \sum_{i=1}^n \langle m, e_i \rangle D_i$. In this way we obtain the map α

$$\alpha : \mathbf{M} \longrightarrow \bigoplus_{i=1}^n \mathbb{Z} \cdot D_i, \quad m \mapsto \sum_{i=1}^n \langle m, e_i \rangle D_i. \quad (15)$$

It follows from the completeness of the fan Σ that the map α is injective. The cokernel of this map defines the Chow group $A_{d-1}(\mathbb{P}_{\Sigma})$ which is a finitely generated abelian group of rank $n - d$. Therefore we have the following exact sequence

$$0 \longrightarrow \mathbf{M} \xrightarrow{\alpha} \bigoplus_{i=1}^n \mathbb{Z} \cdot D_i \xrightarrow{\text{deg}} A_{d-1}(\mathbb{P}_{\Sigma}) \longrightarrow 0, \quad (16)$$

where deg denotes the canonical projection. Now consider $G = \text{Hom}_{\mathbb{Z}}(A_{d-1}(\mathbb{P}_{\Sigma}), \mathbb{C}^*)$ which is in general isomorphic to a product of $(\mathbb{C}^*)^{n-d}$ and a finite group. By applying $\text{Hom}_{\mathbb{Z}}(\cdot, \mathbb{C}^*)$ to (16) we get

$$1 \longrightarrow G \longrightarrow (\mathbb{C}^*)^n \longrightarrow T \longrightarrow 1, \quad (17)$$

which defines the action of G on $\mathbb{C}^n : g \cdot \mathbf{x} = (g(\text{deg}D_i) x_i)$ for $g \in G$ and $\mathbf{x} \in \mathbb{C}^n$.

Let $S = \mathbb{C}[x_1, \dots, x_n]$ be the polynomial ring over \mathbb{C} with variables x_1, \dots, x_n , where x_i correspond to the one dimensional cones ρ_i in Σ . This ring is graded in a natural way by $\text{deg}(x_i) := \text{deg}D_i$. Then, let B be the monomial ideal in S generated by $\prod_{\rho_i \notin \sigma} x_i$ for all $\sigma \in \Sigma$. The ring S defines the n -dimensional affine space $\mathbb{A}^n = \text{Spec}(S)$. The ideal B gives the variety

$$\mathbf{Z}_{\Sigma} = \mathbf{V}(B) \quad (18)$$

which is denoted as the exceptional set. Removing the exceptional set \mathbf{Z}_{Σ} we obtain the Zariski open set

$$\mathbf{U}_{\Sigma} = \mathbb{A}^n \setminus \mathbf{Z}_{\Sigma}, \quad (19)$$

which is invariant under the action of G . For the case of a complete simplicial fan the geometric quotient of \mathbf{U}_Σ by G exists and gives rise to \mathbb{P}_Σ [23]!

Having reviewed these preliminary concepts we now describe the process of resolving the singularities. At first we have to resolve the singularities of the base variety. We begin with a reflexive polytope Δ in \mathbf{N} which corresponds to the weighted projective space $\mathbb{P}(w_1, \dots, w_5)$. Let Σ be the fan in $\mathbf{N}_\mathbb{R}$ associated to Δ . The toric variety \mathbb{P}_Σ then has an ample anticanonical sheaf, whose generic section realizes our (canonical) Calabi-Yau variety.

Taking a maximal triangulation of Δ leads in our case, i.e. $d = 4$, to a fully resolved Calabi-Yau variety \tilde{X} [27]. A maximal triangulation of Δ amounts above all to adding new one dimensional cones to $\Sigma^{(1)}$ which are associated to the points on the faces of Δ . In the context of gauged linear sigma models these correspond to the additional chiral scalar superfields. As mentioned in the previous section we also need to determine the charges of the fields with respect to the full gauge group. Translated into the geometric language this means that we have to determine the grading of the variables in the ring S . Using (15) this can be done by solving

$$\sum_{i=1}^n w_i^{(k)} \alpha_{ij} = 0 \quad \text{for all } k = 1, \dots, N \text{ and } j = 1, \dots, d \quad (20)$$

which gives the charges \mathbf{w}_i of x_i : $\mathbf{w}_i = \deg(x_i) = (w_i^{(1)}, \dots, w_i^{(N)})$. Note that the desingularization of the base variety simultaneously resolves the tangent sheaf to which the right-moving fermions couple. Therefore, these fermions have the same charges as their superpartners.

What about the left-moving fermions? The geometric data of the gauge bundle V in a $(0, 2)$ model are the additional degrees of freedom which we still have to deal with. After resolving the singularities of the base variety we pull the exact sequence (14) back to the desingularized base variety. Because this process only preserves the right exactness of this sequence we are forced to ‘modify’ it in order to get the exact sequence

$$0 \rightarrow \tilde{V} \rightarrow \bigoplus_{i=1}^6 \mathcal{O}(\mathbf{q}_i) \xrightarrow{\tilde{F}} \bigoplus_{j=1}^2 \mathcal{O}(\mathbf{p}_j) \rightarrow 0, \quad (21)$$

where $\mathbf{q}_i = (q_i^{(1)}, \dots, q_i^{(N)})$ and $\mathbf{p}_j = (p_j^{(1)}, \dots, p_j^{(N)})$ denote the degrees (or charges) of the pulled back divisors in the desingularized base variety, which were originally associated to rank one sheaves in (14). Following [18] we impose the same conditions as before on this data guaranteeing the existence of spinors and the cancellation of the gauge anomaly. The conditions $c_1(\widetilde{T_X}) = c_1(\tilde{V}) = 0$ result in

$$\mathbf{d} = \sum_{i=1}^5 \mathbf{w}_i \quad \text{and} \quad \mathbf{p}_1 + \mathbf{p}_2 = \sum_{i=1}^6 \mathbf{q}_i, \quad (22)$$

where $\widetilde{T_X}$ denotes the resolved tangent sheaf and $\mathbf{d} = (d^{(1)}, \dots, d^{(N)})$ is the degree of the pulled back divisor corresponding to the last term in (5). The anomaly cancellation condition leads to the following Diophantine equations

$$d^{(l)}d^{(k)} - \sum_{i=1}^5 w_i^{(l)}w_i^{(k)} = \sum_{j=1}^2 p_j^{(l)}p_j^{(k)} - \sum_{i=1}^6 q_i^{(l)}q_i^{(k)} \quad \text{for } l, k = 1, \dots, N. \quad (23)$$

Each solution of these equations gives a possible gauge bundle data for the desingularized theory. One should be careful about the exactness of (21). It may happen that one can not choose polynomials \widetilde{F} 's such that not all of them vanish simultaneously on the base variety. If this is the case, then one has to deal with a sheaf which is no longer locally free (cf. [18] for details). The examples that will be considered here avoid this problem.

Now we come to the discussion of intersection ring. Let

$$A_*(\mathbb{P}_\Sigma) = \bigoplus_k A_k(\mathbb{P}_\Sigma) \quad (24)$$

be the Chow ring of \mathbb{P}_Σ , where $A_k(\mathbb{P}_\Sigma)$ are the finitely generated abelian groups of k -cycles* in \mathbb{P}_Σ up to rational equivalence. The multiplicative structure of $A_*(\mathbb{P}_\Sigma)$ is given by the intersection of cycles. It is determined by the combinatorial data encoded in the fan Σ . If σ_1 and σ_2 are two cones in Σ that are faces of at least one other cone τ in the fan, then their corresponding cycles do intersect otherwise the intersection is empty. The intersection cycle in the former case is the one that is associated to τ . It is convenient to consider $A^k(\mathbb{P}_\Sigma) := A_{d-k}(\mathbb{P}_\Sigma)$ because by intersecting cycles the codimensions add up. Therefore, we obtain the graded commutative ring

$$A^*(\mathbb{P}_\Sigma) = \bigoplus_k A^k(\mathbb{P}_\Sigma) \quad (25)$$

which is called the intersection ring of \mathbb{P}_Σ . Let $\mathbb{Q}[z_1, \dots, z_n]$ be the polynomial ring in variables z_1, \dots, z_n over \mathbb{Q} , where z_i correspond to the one dimensional cones ρ_i in the fan Σ . Further, let $I = \langle \sum_{i=1}^n \langle m, e_i \rangle z_i : m \in \mathbf{M} \rangle$ and $J = \langle \prod_{\rho_i \in \mathcal{P}} z_i : \text{for all } \mathcal{P} \rangle$ (is called Stanley–Reisner ideal) be ideals in $\mathbb{Q}[z_1, \dots, z_n]$, where \mathcal{P} stands for a primitive collection. It is a subset $\{\rho_{i_1}, \dots, \rho_{i_k}\}$ of $\Sigma^{(1)}$ which does not generate a k -dimensional cone, whereas any proper subset of it generates a cone in Σ . It can be shown that for a complete simplicial toric variety \mathbb{P}_Σ one has the following isomorphisms for the intersection ring $A^*(\mathbb{P}_\Sigma)_\mathbb{Q} (= A^*(\mathbb{P}_\Sigma) \otimes_\mathbb{Z} \mathbb{Q})$

$$A^*(\mathbb{P}_\Sigma)_\mathbb{Q} \simeq \mathbb{Q}[z_1, \dots, z_n]/(I + J) \simeq H^*(\mathbb{P}_\Sigma, \mathbb{Q}), \quad (26)$$

where the isomorphism in the direction of the rational cohomology ring doubles the degree. We are actually interested in the intersection ring of the desingularized Calabi-Yau variety \widetilde{X} which, as pointed out above, is a generic section of the anticanonical sheaf on \mathbb{P}_Σ . The

*These are the closed T -orbits associated to the elements of $\Sigma^{(d-k)}$.

ring $A^*(\widetilde{X})_{\mathbb{Q}}$ is isomorphic to the quotient of $A^*(\mathbb{P}_{\Sigma})_{\mathbb{Q}}$ by the annihilator of the canonical divisor [33], i.e.

$$A^*(\widetilde{X})_{\mathbb{Q}} \simeq A^*(\mathbb{P}_{\Sigma})_{\mathbb{Q}} / \text{ann}(z_1 + \dots + z_n) . \quad (27)$$

Let \mathcal{F} be a coherent sheaf on the (complete) variety \mathbf{X} . As we know, the Euler characteristic of \mathcal{F} , $\chi(\mathbf{X}, \mathcal{F})$, is defined by

$$\chi(\mathcal{F}) = \chi(\mathbf{X}, \mathcal{F}) = \sum_{p \geq 0} (-1)^p H^p(\mathbf{X}, \mathcal{F}) . \quad (28)$$

The main property of the Euler characteristic is its additivity. It means that for an exact sequence of coherent sheaves $0 \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow 0$ it holds that $\chi(\mathcal{F}) = \chi(\mathcal{E}) + \chi(\mathcal{G})$.

The Riemann-Roch theorem allows us to express the Euler characteristic of a coherent sheaf \mathcal{F} on the (smooth) variety \mathbf{X} in terms of the intersection of algebraic cycles in \mathbf{X} . We now define the intersection form (\cdot, \cdot) on $A^*(\mathbf{X})$ which we need in the statement of the Riemann-Roch theorem. It is a map $A^*(\mathbf{X}) \times A^*(\mathbf{X}) \rightarrow \mathbb{Q}$ that is the composition of the multiplication in $A^*(\mathbf{X})$ and the linear functional $\text{deg} : A^*(\mathbf{X}) \rightarrow \mathbb{Q}$ defined as follows. deg associates to each cycle $\eta \in A^*(\mathbf{X})$ the degree of its 0-cycle part $\eta_0 = \sum_i n_i [P_i] : \text{deg}(\eta_0) = \sum_i n_i$. For a smooth projective variety \mathbf{X} the Riemann-Roch theorem reads

$$\chi(\mathcal{F}) = (\text{ch}(\mathcal{F}), \text{Td}(\mathbf{X})) , \quad (29)$$

where $\text{ch}(\mathcal{F})$ is the Chern character of \mathcal{F} and $\text{Td}(\mathbf{X}) = \text{td}(T_{\mathbf{X}})$ is the Todd class of the tangent sheaf. $\text{ch}(\mathcal{F})$ and $\text{Td}(\mathbf{X})$ are elements of $A^*(\mathbf{X})$! We recall that the Chern character of a coherent sheaf \mathcal{F} is defined through its locally free resolution. For a locally free sheaf \mathcal{E} of rank r we have:

$$\begin{aligned} \text{ch}(\mathcal{E}) &= r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) \\ &\quad + \frac{1}{24}(c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4) + \dots \end{aligned} \quad (30)$$

$$\begin{aligned} \text{td}(\mathcal{E}) &= 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 \\ &\quad - \frac{1}{720}(c_1^4 - 4c_1^2c_2 - 3c_2^2 - c_1c_3 + c_4) + \dots , \end{aligned} \quad (31)$$

where $c_i = c_i(\mathcal{E})$ is the i -th Chern class of \mathcal{E} .

4 Examples

In this section we discuss a few examples of the class of models given by the geometric data of (14) on a Calabi-Yau variety in the weighted projective space $\mathbb{P}(w_1, \dots, w_5)$.

Example 1 : $\mathbb{P}(1, 1, 1, 3, 3)$

The gauge bundle V is given by

$$0 \rightarrow V \rightarrow \mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}(3) \oplus \mathcal{O}(6) \oplus \mathcal{O}(3) \rightarrow \mathcal{O}(6) \oplus \mathcal{O}(9) \rightarrow 0 .$$

The reflexive polytope Δ corresponding to $\mathbb{P}(1, 1, 1, 3, 3)$ is defined by the vertices

$$\begin{aligned} e_1 &= (1, 0, 0, 0), e_2 = (0, 1, 0, 0), e_3 = (0, 0, 1, 0) \\ e_4 &= (0, 0, 0, 1), e_5 = (-1, -1, -3, -3) \end{aligned}$$

with respect to canonical basis in the lattice \mathbf{N} . Apart from the unique inner point, Δ still has one additional point e_6 which lies on the codimension 2 face of Δ generated as the convex hull of the points e_1, e_2 and e_5 : $e_6 = (0, 0, -1, -1)$. Therefore, the desingularization in this case gives rise to an extra $U(1)$ factor. We now proceed to find the charges of the fields. As before we denote by x_i and D_i the variables in S and the divisors associated to e_i . In the canonical basis $\{u_i\}_{i=1}^4$ of \mathbf{M} , the map α is represented by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -3 & -3 \\ 0 & 0 & -1 & -1 \end{pmatrix} ,$$

which yields

field	x_1	x_2	x_3	x_4	x_5	x_6
charge	$(1, 0)$	$(1, 0)$	$(3, 1)$	$(3, 1)$	$(1, 0)$	$(0, 1)$

Using this table we determine in the next step the data of the resolved gauge bundle \tilde{V} . From (22) and (23) we obtain

$$\begin{aligned} q_1 + q_2 + q_3 + q_4 + q_5 + q_6 - p_1 - p_2 &= 0 , \\ q_1 + q_2 + q_3 + 3q_4 + 6q_5 + 3q_6 - 6p_1 - 9p_2 &= -12 , \\ q_1^2 + q_2^2 + q_3^2 + q_4^2 + q_5^2 + q_6^2 - p_1^2 - p_2^2 &= -2 , \end{aligned}$$

where we have dropped the index (2) on q 's and p 's. Here is a set of solutions of the above Diophantine equations

q_1	q_2	q_3	q_4	q_5	q_6	p_1	p_2
0	-1	1	1	4	0	2	3
0	0	0	1	0	1	0	2
1	1	0	0	0	0	2	0
0	0	0	1	1	1	1	2

which result in

(a)	$0 \rightarrow \tilde{V} \rightarrow \mathcal{O}(1, 0) \oplus \mathcal{O}(1, -1) \oplus \mathcal{O}(1, 1) \oplus \mathcal{O}(6, 4)$ $\oplus \mathcal{O}(3, 1) \oplus \mathcal{O}(3, 0) \rightarrow \mathcal{O}(6, 2) \oplus \mathcal{O}(9, 3) \rightarrow 0$
(b)	$0 \rightarrow \tilde{V} \rightarrow \mathcal{O}(1, 0) \oplus \mathcal{O}(1, 0) \oplus \mathcal{O}(1, 0) \oplus \mathcal{O}(3, 1)$ $\oplus \mathcal{O}(6, 0) \oplus \mathcal{O}(3, 1) \rightarrow \mathcal{O}(6, 0) \oplus \mathcal{O}(9, 2) \rightarrow 0$
(c)	$0 \rightarrow \tilde{V} \rightarrow \mathcal{O}(1, 1) \oplus \mathcal{O}(1, 1) \oplus \mathcal{O}(1, 0) \oplus \mathcal{O}(3, 0)$ $\oplus \mathcal{O}(6, 0) \oplus \mathcal{O}(3, 0) \rightarrow \mathcal{O}(6, 2) \oplus \mathcal{O}(9, 0) \rightarrow 0$
(d)	$0 \rightarrow \tilde{V} \rightarrow \mathcal{O}(1, 0) \oplus \mathcal{O}(1, 0) \oplus \mathcal{O}(1, 0) \oplus \mathcal{O}(3, 1)$ $\oplus \mathcal{O}(6, 1) \oplus \mathcal{O}(3, 1) \rightarrow \mathcal{O}(6, 1) \oplus \mathcal{O}(9, 2) \rightarrow 0$

We now consider a fan Σ corresponding to the maximal triangulation of Δ given by the ‘big cones’

$$\begin{aligned}
&\langle e_1 e_2 e_3 e_4 \rangle, \langle e_1 e_2 e_3 e_6 \rangle, \langle e_1 e_2 e_4 e_6 \rangle, \\
&\langle e_1 e_3 e_4 e_5 \rangle, \langle e_1 e_3 e_5 e_6 \rangle, \langle e_1 e_4 e_5 e_6 \rangle, \\
&\langle e_2 e_3 e_4 e_5 \rangle, \langle e_2 e_3 e_5 e_6 \rangle, \langle e_2 e_4 e_5 e_6 \rangle,
\end{aligned}$$

where $\langle e_i e_j e_k e_l \rangle$ denotes the cone generated by e_i, e_j, e_k and e_l . The primitive collections of Σ are $\{e_3, e_4, e_6\}$ and $\{e_1, e_2, e_5\}$. With these combinatorial data at hand we can write down the ideals I and J :

$$I = \langle z_1 - z_5, z_2 - z_5, z_3 - 3z_5 - z_6, z_4 - 3z_5 - z_6 \rangle, \quad J = \langle z_1 z_2 z_5, z_3 z_4 z_6 \rangle.$$

Because we are going to make calculations in a polynomial ring, it is convenient to use the Gröbner basis method [34, 35, 36]. A Gröbner basis of $I + J$ with respect to the lex order $z_1 > \dots > z_6$ is given by

$$\begin{aligned}
I + J = \langle &z_1 - z_5, z_2 - z_5, z_3 - 3z_5 - z_6, z_4 - 3z_5 - z_6, \\
&z_5^3, 9z_5^2 z_6 + 6z_5 z_6^2 + z_6^3, 9z_5 z_6^3 + 2z_6^4, z_6^5 \rangle.
\end{aligned}$$

Let K be the ideal in the polynomial ring $\mathbb{Q}[z_1, \dots, z_6]$ generated by $z_1 + \dots + z_6$ which is a representative of the canonical class in the intersection ring $A^*(\mathbb{P}_\Sigma)$. Then the annihilator of $z_1 + \dots + z_6$ in $A^*(\mathbb{P}_\Sigma)$ is given by

$$\text{ann}(z_1 + \dots + z_6) = (I + J) : K,$$

where ‘:’ denotes the quotient of ideals. $\text{ann}(z_1 + \dots + z_6)$ in this example is calculated to

$$\text{ann}(z_1 + \dots + z_6) = \langle z_1 + z_2 + z_3 + z_4 - 8z_5 - 2z_6, z_2 - z_5, \\ z_3 - 3z_5 - z_6, z_4 - 3z_5 - z_6, z_5^3, 3z_5z_6 + z_6^2, z_6^4 \rangle.$$

A look at (30) and (31) shows that in the product of $\text{ch}(\tilde{V})$ and $\text{Td}(\tilde{X})$ in the intersection ring $A^*(\tilde{X})$ the only 0-cycle part is given by $1/2 c_3(\tilde{V})$. Applying the degree functional to this term gives us the Euler characteristic of the respective gauge bundle. One should be careful about the normalization of the product of cycles. For the big cones of Σ the normalization is fixed by

$$\langle D_{i_1} \dots D_{i_4} \rangle = \frac{1}{\text{mult}(e_{i_1}, \dots, e_{i_4})}, \quad (32)$$

where $\text{mult}(e_{i_1}, \dots, e_{i_4})$ denotes the index in \mathbf{N} of the lattice spanned by these vectors. Multiplying the top terms in the intersection ring of the desingularized Calabi-Yau variety by the representative of the canonical divisor and using (32) together with the ‘algebraic moving lemma’ [20, 21] yields the normalization in $A^*(\tilde{X})$.

All big cones in this example have volume one. Therefore, $\langle D_{i_1} \dots D_{i_4} \rangle = 1$ for all big cones in Σ . As we will see below, the third Chern class of the gauge bundle is represented by a degree three monomial in z_6 . Its normalization in $A^*(\tilde{X})$ is given by $\langle z_6^3 \rangle = 27$. We have summarized the result of the calculations for the resolved bundles found above in the following table

\tilde{V}	$c_3(\tilde{V})$	$\chi(\tilde{V})$
(a)	$-\frac{20}{3}z_6^3$	−90
(b)	$-8z_6^3$	−108
(c)	$-\frac{22}{9}z_6^3$	−33
(d)	$-8z_6^3$	−108

Example 2 : $\mathbb{P}(1, 1, 2, 2, 3)$

The gauge bundle V is given by

$$0 \rightarrow V \rightarrow \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}(2)^{\oplus 2} \oplus \mathcal{O}(6) \oplus \mathcal{O}(3) \rightarrow \mathcal{O}(6) \oplus \mathcal{O}(9) \rightarrow 0.$$

The reflexive polytope Δ corresponding to $\mathbb{P}(1, 1, 2, 2, 3)$ is defined by the vertices

$$\begin{aligned} e_1 &= (1, 0, 0, 0), e_2 = (0, 1, 0, 0), e_3 = (0, 0, 1, 0) \\ e_4 &= (0, 0, 0, 1), e_5 = (-1, -2, -2, -3), e_6 = (0, -1, -1, -1) . \end{aligned}$$

This exhausts the set of boundary points of Δ . The desingularization gives rise as before to an extra $U(1)$ factor. The map α is represented by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -2 & -2 & -3 \\ 0 & -1 & -1 & -1 \end{pmatrix} ,$$

which yields

field	x_1	x_2	x_3	x_4	x_5	x_6
charge	(1, 0)	(2, 1)	(2, 1)	(3, 1)	(1, 0)	(0, 1)

Using this table we obtain the following equations for the data of the resolved gauge bundle \tilde{V}

$$\begin{aligned} q_1 + q_2 + q_3 + q_4 + q_5 + q_6 - p_1 - p_2 &= 0 , \\ q_1 + q_2 + 2q_3 + 2q_4 + 6q_5 + 3q_6 - 6p_1 - 9p_2 &= -20 , \\ q_1^2 + q_2^2 + q_3^2 + q_4^2 + q_5^2 + q_6^2 - p_1^2 - p_2^2 &= -6 . \end{aligned}$$

Two solutions of these equations are

q_1	q_2	q_3	q_4	q_5	q_6	p_1	p_2
1	1	1	0	2	0	3	2
2	0	1	0	3	0	4	2

which lead to

(a)	$0 \rightarrow \tilde{V} \rightarrow \mathcal{O}(1, 1) \oplus \mathcal{O}(1, 1) \oplus \mathcal{O}(2, 1) \oplus \mathcal{O}(2, 0)$ $\oplus \mathcal{O}(6, 2) \oplus \mathcal{O}(3, 0) \rightarrow \mathcal{O}(6, 3) \oplus \mathcal{O}(9, 2) \rightarrow 0$
(b)	$0 \rightarrow \tilde{V} \rightarrow \mathcal{O}(1, 2) \oplus \mathcal{O}(1, 0) \oplus \mathcal{O}(2, 1) \oplus \mathcal{O}(2, 0)$ $\oplus \mathcal{O}(6, 3) \oplus \mathcal{O}(3, 0) \rightarrow \mathcal{O}(6, 4) \oplus \mathcal{O}(9, 2) \rightarrow 0$

The big cones of the fan Σ corresponding to the maximal triangulation of Δ are

$$\begin{aligned} &\langle e_1 e_2 e_3 e_4 \rangle, \langle e_1 e_2 e_3 e_5 \rangle, \langle e_1 e_2 e_4 e_6 \rangle, \\ &\langle e_1 e_2 e_5 e_6 \rangle, \langle e_1 e_3 e_4 e_6 \rangle, \langle e_1 e_3 e_5 e_6 \rangle, \\ &\langle e_2 e_3 e_4 e_5 \rangle, \langle e_2 e_4 e_5 e_6 \rangle, \langle e_3 e_4 e_5 e_6 \rangle. \end{aligned}$$

The primitive collections of Σ are $\{e_2, e_3, e_6\}$ and $\{e_1, e_4, e_5\}$. Using these combinatorial data we find $I = \langle z_1 - z_5, z_2 - 2z_5 - z_6, z_3 - 2z_5 - z_6, z_4 - 3z_5 - z_6 \rangle$ and $J = \langle z_1 z_4 z_5, z_2 z_3 z_6 \rangle$. With respect to the lex order $z_1 > \dots > z_6$ the Gröbner bases of $I + J$ and $\text{ann}(z_1 + \dots + z_6)$ are given by

$$\begin{aligned} I + J &= \langle z_1 - z_5, z_2 - 2z_5 - z_6, z_3 - 2z_5 - z_6, z_4 - 3z_5 - z_6, \\ &\quad 3z_5^3 + z_5^2 z_6, 4z_5^2 z_6 + 4z_5 z_6^2 + z_6^3, 5z_5 z_6^3 + 2z_6^4, z_6^5 \rangle \\ \text{ann}(z_1 + \dots + z_6) &= \langle z_1 + z_2 + z_3 + z_4 - 8z_5 - 3z_6, z_2 - 2z_5 - z_6, z_3 - 2z_5 - z_6, \\ &\quad z_4 - 3z_5 - z_6, z_5^2 + 7z_5 z_6 + 4z_6^2, 8z_5 z_6^2 + 5z_6^3, z_6^4 \rangle. \end{aligned}$$

The normalization in this case is as follows: $\langle D_1 D_2 D_3 D_5 \rangle = \frac{1}{3}$ and all other big cones have unit volume. This leads to the normalization $\langle z_6^3 \rangle = 8$ in the intersection ring $A^*(\widetilde{X})$. Therefore, we obtain

\tilde{V}	$c_3(\tilde{V})$	$\chi(\tilde{V})$
(a)	$-\frac{51}{4}z_6^3$	-51
(b)	$-\frac{21}{2}z_6^3$	-42

Example 3 : $\mathbb{P}(1, 2, 2, 3, 4)$

The gauge bundle V is given by

$$0 \rightarrow V \rightarrow \mathcal{O}(1) \oplus \mathcal{O}(2)^{\oplus 2} \oplus \mathcal{O}(3) \oplus \mathcal{O}(8) \oplus \mathcal{O}(4) \rightarrow \mathcal{O}(8) \oplus \mathcal{O}(12) \rightarrow 0.$$

The reflexive polytope Δ corresponding to $\mathbb{P}(1, 2, 2, 3, 4)$ is defined by the vertices

$$\begin{aligned} e_1 &= (1, 0, 0, 0), e_2 = (0, 1, 0, 0), e_3 = (0, 0, 1, 0) \\ e_4 &= (0, 0, 0, 1), e_5 = (-2, -2, -3, -4). \end{aligned}$$

Δ still has one other boundary point $e_6 = (-1, -1, -1, -2)$ which lies on the codimension three face generated by e_3 and e_5 . The desingularization gives rise as before to an extra

$U(1)$ factor. The map α is represented by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -2 & -3 & -4 \\ -1 & -1 & -1 & -2 \end{pmatrix},$$

which yields

field	x_1	x_2	x_3	x_4	x_5	x_6
charge	$(2, 1)$	$(2, 1)$	$(3, 1)$	$(4, 2)$	$(1, 0)$	$(0, 1)$

Therefore, we obtain the following equations for the data of the resolved gauge bundle \tilde{V}

$$\begin{aligned} q_1 + q_2 + q_3 + q_4 + q_5 + q_6 - p_1 - p_2 &= 0, \\ q_1 + 2q_2 + 2q_3 + 3q_4 + 8q_5 + 4q_6 - 8p_1 - 12p_2 &= -45, \\ q_1^2 + q_2^2 + q_3^2 + q_4^2 + q_5^2 + q_6^2 - p_1^2 - p_2^2 &= -18. \end{aligned}$$

Two solutions of these equations are

q_1	q_2	q_3	q_4	q_5	q_6	p_1	p_2
1	1	1	2	0	2	5	2
2	0	2	3	0	0	6	1

which lead to

(a)	$0 \rightarrow \tilde{V} \rightarrow \mathcal{O}(1, 1) \oplus \mathcal{O}(2, 1) \oplus \mathcal{O}(2, 1) \oplus \mathcal{O}(3, 2)$ $\oplus \mathcal{O}(8, 0) \oplus \mathcal{O}(4, 2) \rightarrow \mathcal{O}(8, 5) \oplus \mathcal{O}(12, 2) \rightarrow 0$
(b)	$0 \rightarrow \tilde{V} \rightarrow \mathcal{O}(1, 2) \oplus \mathcal{O}(2, 0) \oplus \mathcal{O}(2, 2) \oplus \mathcal{O}(3, 3)$ $\oplus \mathcal{O}(8, 0) \oplus \mathcal{O}(4, 0) \rightarrow \mathcal{O}(8, 6) \oplus \mathcal{O}(12, 1) \rightarrow 0$

The big cones of the fan Σ corresponding to the maximal triangulation of Δ are

$$\begin{aligned} \langle e_1 e_2 e_3 e_4 \rangle, \langle e_1 e_2 e_3 e_6 \rangle, \langle e_1 e_2 e_4 e_5 \rangle, \\ \langle e_1 e_2 e_5 e_6 \rangle, \langle e_1 e_3 e_4 e_6 \rangle, \langle e_1 e_4 e_5 e_6 \rangle, \\ \langle e_2 e_3 e_4 e_6 \rangle, \langle e_2 e_4 e_5 e_6 \rangle. \end{aligned}$$

The primitive collections of Σ are $\{e_3, e_5\}$ and $\{e_1, e_2, e_4, e_6\}$. Using these combinatorial data we find $I = \langle z_1 - 2z_5 - z_6, z_2 - 2z_5 - z_6, z_3 - 3z_5 - z_6, z_4 - 4z_5 - 2z_6 \rangle$ and $J = \langle z_3z_5, z_1z_2z_4z_6 \rangle$. With respect to the lex order $z_1 > \dots > z_6$ the Gröbner bases of $I + J$ and $\text{ann}(z_1 + \dots + z_6)$ are given by

$$\begin{aligned} I + J &= \langle z_1 - 2z_5 - z_6, z_2 - 2z_5 - z_6, z_3 - 3z_5 - z_6, \\ &\quad z_4 - 4z_5 - 2z_6, 3z_5^2 + z_5z_6, 26z_5z_6^3 + 9z_6^4, z_6^5 \rangle \\ \text{ann}(z_1 + \dots + z_6) &= \langle z_1 + z_2 + z_3 + z_4 - 11z_5 - 5z_6, z_2 - 2z_5 - z_6, z_3 - 3z_5 - z_6, \\ &\quad z_4 - 4z_5 - 2z_6, 3z_5^2 + z_5z_6, 8z_5z_6^2 + 3z_6^3, z_6^4 \rangle. \end{aligned}$$

The normalization in this case is as follows:

$$\langle D_1D_2D_3D_6 \rangle = \langle D_1D_2D_5D_6 \rangle = \frac{1}{2}, \quad \langle D_1D_2D_4D_5 \rangle = \frac{1}{3},$$

and all other big cones have unit volume. This results in the normalization $\langle z_6^3 \rangle = -24$ in the intersection ring $A^*(\tilde{X})$. Therefore, we obtain

\tilde{V}	$c_3(\tilde{V})$	$\chi(\tilde{V})$
(a)	$-\frac{7}{2}z_6^3$	42
(b)	z_6^3	-12

Example 4 : $\mathbb{P}(1, 1, 3, 3, 4)$

The gauge bundle V is given by

$$0 \rightarrow V \rightarrow \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}(3)^{\oplus 2} \oplus \mathcal{O}(8) \oplus \mathcal{O}(4) \rightarrow \mathcal{O}(8) \oplus \mathcal{O}(12) \rightarrow 0,$$

and the vertices

$$\begin{aligned} e_1 &= (1, 0, 0, 0), e_2 = (0, 1, 0, 0), e_3 = (0, 0, 1, 0) \\ e_4 &= (0, 0, 0, 1), e_5 = (-1, -3, -3, -4). \end{aligned}$$

define the reflexive polytope Δ corresponding to $\mathbb{P}(1, 1, 3, 3, 4)$. There still exists one other boundary point $e_6 = (0, -1, -1, -1)$ of Δ which lies on the codimension two face spanned by e_1, e_4 and e_5 . An extra $U(1)$ factor arises from the desingularization. The map α is represented by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -3 & -3 & -4 \\ 0 & -1 & -1 & -1 \end{pmatrix},$$

which yields

field	x_1	x_2	x_3	x_4	x_5	x_6
charge	$(1, 0)$	$(3, 1)$	$(3, 1)$	$(4, 1)$	$(1, 0)$	$(0, 1)$

This leads to the following equations for the data of the resolved gauge bundle \tilde{V}

$$\begin{aligned}
q_1 + q_2 + q_3 + q_4 + q_5 + q_6 - p_1 - p_2 &= 0 , \\
q_1 + q_2 + 3q_3 + 3q_4 + 8q_5 + 4q_6 - 8p_1 - 12p_2 &= -26 , \\
q_1^2 + q_2^2 + q_3^2 + q_4^2 + q_5^2 + q_6^2 - p_1^2 - p_2^2 &= -6 .
\end{aligned}$$

Two solutions of these equations are

q_1	q_2	q_3	q_4	q_5	q_6	p_1	p_2
1	1	0	0	2	1	3	2
2	0	0	0	3	1	4	2

which result in

(a)	$0 \rightarrow \tilde{V} \rightarrow \mathcal{O}(1, 1) \oplus \mathcal{O}(1, 1) \oplus \mathcal{O}(3, 0) \oplus \mathcal{O}(3, 0)$ $\oplus \mathcal{O}(8, 2) \oplus \mathcal{O}(4, 1) \rightarrow \mathcal{O}(8, 3) \oplus \mathcal{O}(12, 2) \rightarrow 0$
(b)	$0 \rightarrow \tilde{V} \rightarrow \mathcal{O}(1, 2) \oplus \mathcal{O}(1, 0) \oplus \mathcal{O}(3, 0) \oplus \mathcal{O}(3, 0)$ $\oplus \mathcal{O}(8, 3) \oplus \mathcal{O}(4, 1) \rightarrow \mathcal{O}(8, 4) \oplus \mathcal{O}(12, 2) \rightarrow 0$

The big cones of the fan Σ corresponding to the maximal triangulation of Δ are

$$\begin{aligned}
&\langle e_1 e_2 e_3 e_4 \rangle, \langle e_1 e_2 e_3 e_5 \rangle, \langle e_1 e_2 e_4 e_6 \rangle, \\
&\langle e_1 e_2 e_5 e_6 \rangle, \langle e_1 e_3 e_4 e_6 \rangle, \langle e_1 e_3 e_5 e_6 \rangle, \\
&\langle e_2 e_3 e_4 e_5 \rangle, \langle e_2 e_4 e_5 e_6 \rangle, \langle e_3 e_4 e_5 e_6 \rangle .
\end{aligned}$$

The primitive collections of Σ are $\{e_2, e_3, e_6\}$ and $\{e_1, e_4, e_5\}$. From these combinatorial data we find $I = \langle z_1 - z_5, z_2 - 3z_5 - z_6, z_3 - 3z_5 - z_6, z_4 - 4z_5 - z_6 \rangle$ and $J = \langle z_1 z_4 z_5, z_2 z_3 z_6 \rangle$. With respect to the lex order $z_1 > \dots > z_6$ the Gröbner bases of $I + J$ and $\text{ann}(z_1 + \dots + z_6)$ are given by

$$\begin{aligned}
I + J &= \langle z_1 - z_5, z_2 - 3z_5 - z_6, z_3 - 3z_5 - z_6, z_4 - 4z_5 - z_6, \\
&\quad 4z_5^3 + z_5^2 z_6, 9z_5^2 z_6 + 6z_5 z_6^2 + z_6^3, 18z_5 z_6^3 + 5z_6^4, z_6^5 \rangle \\
\text{ann}(z_1 + \dots + z_6) &= \langle z_1 + z_2 + z_3 + z_4 - 11z_5 - 3z_6, z_2 - 3z_5 - z_6, z_3 \\
&\quad - 3z_5 - z_6, z_4 - 4z_5 - z_6, 4z_5^3 + z_5^2 z_6, 3z_5 z_6 + z_6^2, z_6^4 \rangle .
\end{aligned}$$

The normalization in this case is as follows: $\langle D_1 D_2 D_3 D_5 \rangle = \frac{1}{4}$ and all other big cones have unit volume. This leads to the normalization $\langle z_6^3 \rangle = 36$ in the intersection ring $A^*(\widetilde{X})$. Therefore, we obtain

\widetilde{V}	$c_3(\widetilde{V})$	$\chi(\widetilde{V})$
(a)	$-\frac{26}{9}z_6^3$	-52
(b)	$-\frac{20}{9}z_6^3$	-40

5 Conclusion

Starting from a series of solutions of the anomaly cancellation equation we have constructed a class of $(0, 2)$ Calabi-Yau σ models. These solutions are associated to certain $(0, 2)$ Landau-Ginzburg models which are conjectured to be ‘equivalent’ to the $(0, 2)$ superconformal field theories constructed in [15, 16, 17]. Following [18] we have studied the desingularization of a few examples from this class. This led in each case to a family of $(0, 2)$ Calabi-Yau σ models. As pointed out above, the ambiguity in the geometric interpretation of the $(0, 2)$ models has, in contrast to the $(2, 2)$ case, two different sources. The first one is, as in $(2, 2)$ models, the choice of maximal triangulations of the reflexive polytope Δ [32, 29]. The second one comes from the different ways of ‘pulling the gauge bundle back to the desingularized Calabi-Yau variety’. It seems to be natural to ask if there exists a selection rule which associate to a given superconformal field theory a subset of the desingularized Calabi-Yau σ models as its possible geometric realizations. The explicit knowledge of the exact superconformal theories in our case will be useful in answering this question for the class of models considered here. Another issue that can be addressed is the following. As mentioned above, the solution of the anomaly cancellation equation can be equally interpreted as the defining data of a bundle on a Calabi-Yau complete intersection. (To deal with this latter case is, however, technically more cumbersome.) It would be interesting to study the consequences of this dual interpretation of the anomaly cancellation condition and to compare the desingularized models in these two cases.

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Appendix : Gröbner basis

Let $k[x_1, \dots, x_m]$ denote a polynomial ring in m variables x_1, \dots, x_m over the field k . To each monomial $x^\alpha = x_1^{\alpha_1} \dots x_m^{\alpha_m}$ we associate the element $(\alpha_1, \dots, \alpha_m)$ in the semigroup $(\mathbb{N}^m, +)$. By a *monomial ordering* in $k[x_1, \dots, x_m]$ we mean an order relation on the set of monomials induced by a total ordering $>$ on \mathbb{N}^m which is consistent with its semi-group structure and such that $>$ is a well-ordering on \mathbb{N}^m . Here are two examples: (1) *lex(icographic) order*: $x^\alpha >_{lex} x^\beta \Leftrightarrow$ (the left-most nonzero entry in $\alpha - \beta$ is positive), (2) *g(raded) lex order*: $x^\alpha >_{glex} x^\beta \Leftrightarrow (\sum_i \alpha_i > \sum_i \beta_i)$ or $(\alpha = \beta \text{ and } x^\alpha >_{lex} x^\beta)$. Given a nonzero polynomial $f = \sum_\alpha a_\alpha x^\alpha$ in $k[x_1, \dots, x_m]$ with a monomial order we define: $\deg(f) = \alpha_{max} = \max(\alpha \in \mathbb{N}^m : a_\alpha \neq 0)$, leading term of $f = lt(f) = a_{\alpha_{max}} x^{\alpha_{max}}$, leading monomial of $f = lm(f) = x^{\alpha_{max}}$. We now come to the division algorithm.

Division algorithm in $k[x_1, \dots, x_m]$

input: a s -tuple of polynomials $F = (f_1, \dots, f_s)$ and a nonzero polynomial f ,
output: the remainder $r (= \bar{f}^F)$ of dividing f by F and the quotients q_1, \dots, q_s ,
algorithm: $p := f$, $r := 0$, $q_i := 0$ for all $i = 1, \dots, s$

repeat

$i := 1$, dividing:=true

while $(i \leq s)$ and (dividing) **do**

if $lt(f_i)$ divides $lt(p)$ **then**

$u := lt(p)/lt(f_i)$, $q_i := q_i + u$, $p := p - u f_i$, dividing:=false

else $i := i + 1$

if dividing **then**

$r := r + lt(p)$, $p := p - lt(p)$

until $p = 0$

It should be noted that, for a given monomial order, r and q_i depened on the order of f_i in F . Given $f, g \in k[x_1, \dots, x_m]$ with $h = \text{LCM}(lm(f), lm(g))$ we define the S -polynomial of f and g as $S(f, g) = h \cdot (f/lt(f) - g/lt(g))$. Now let I be an ideal in $k[x_1, \dots, x_m]$. A *Gröbner basis* of I is a generating set $G = \{f_1, \dots, f_s\}$ such that $\overline{S(f_i, f_j)}^G = 0$ for all i and j . The remainder of dividing a polynomial by G is unique! Using the *Buchberger's algorithm* one can find a Gröbner basis of a given ideal.

Buchberger's algorithm

input: a s -tuple of polynomials $F = (f_1, \dots, f_r)$ which generates I ,
output: a Gröbner basis $G = (g_1, \dots, g_s)$ of I ,
algorithm: $G := F$

repeat

$G' := G$

for each i, j with $i \neq j$ in G' **do**

$S := \overline{S(f_i, f_j)}^{G'}$

if $S \neq 0$ **then** $G := G \cup \{S\}$

until $G = G'$

Using the Gröbner basis we can do algorithmic calculations in a polynomial ring. As an example we give the algorithm for the calculation of $I : J = \{f \in k[x_1, \dots, x_m] \mid fJ \subset I\}$. First we determine a Gröbner basis of $I \cap J$. It is given as the intersection of $k[x_1, \dots, x_m]$ with a Gröbner basis of the ideal $tI - (1 - t)J$ in $k[t, x_1, \dots, x_m]$ with respect to a lex order in which t is greater than x_i . Let $J = \langle f_1, \dots, f_n \rangle$. Taking $I : J = I : \langle f_1, \dots, f_n \rangle = \bigcap_{i=1}^n I : \langle f_i \rangle$ into account we only need to calculate a Gröbner basis of $I : \langle f_i \rangle$. Because of $I : \langle f_i \rangle = 1/f(I \cap \langle f_i \rangle)$ it reduces to the case just discussed above (cf. [34, 35, 36] for more details).

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